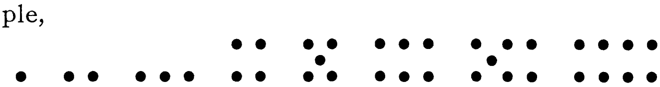
NOTION AND DEFINITION OF NUMBER.

MANY essays have been written on the definition of number. But most of them contain too many technical expressions, philosophical and mathematical, to meet the taste of the non-mathematician. The clearest idea of what counting and numbers mean may be gained from the observation of children and of nations in the childhood of civilisation. When children count or add, they use either their fingers, or small sticks of wood, or pebbles, or similar things, which they separately adjoin to the things to be counted or otherwise ordinally associate with them. As we know from history, the Romans and Greeks employed their fingers when they counted or added. And even to-day we frequently meet with people to whom the use of the fingers is absolutely indispensable for computation.

Still better proof that the accurate association of such "other" things with the things to be counted is the essential element of numeration are the tales of travellers in Africa, telling us how African tribes sometimes inform friendly nations of the number of the enemies who have invaded their domain. The conveyance of the information is effected not by messengers, but simply by placing at spots selected for the purpose a number of stones exactly equal to the number of the invaders. No one will deny that the number of the tribe's foes is thus communicated, even though no name exists for this number in the languages of the tribes. The reason why the fingers are so universally employed as a means of numeration is, that every one possesses a definite number of fingers, sufficiently large for purposes of computation and that they are always at hand.

Besides this first and chief element of numeration which, as we

have seen, is the exact, individual conjunction or association of other things with the things to be counted, is to be mentioned a second important element, which in some respects perhaps is not so absolutely essential, namely, that the things to be counted shall be regarded as of the same kind ; thus, any one who subjects apples and nuts collectively to a process of numeration will regard them for the time being as objects of the same kind, perhaps, by subsuming them under the common notion of fruit. We may therefore lay down provisionally the following as a definition of counting : to count a group of things is to regard the things as the same in kind and to associate ordinally, accurately, and singly with them other things. In writing, we associate with the things to be counted simple signs, like points, strokes, or circles. The form of the symbols we use is indifferent. Neither need they be uniform. It is also indifferent what the spatial relations or dispositions of these symbols are. Although, of course, it is much more convenient and simpler to fashion symbols growing out of operations of counting on principles of uniformity and to place them spatially near each other. In this manner are produced what I have called* natural number-pictures ; for example,


etc.

Now-a-days such natural number-pictures are rarely employed, and are to be seen only on dominoes, dice, and sometimes, also, on playing-cards.

It can be shown by archæological evidence that originally numeral writing was made up wholly of natural number-pictures. For example, the Romans in early times represented all numbers, which were written at all, by assemblages of strokes. We have remnants of this writing in the first three numerals of the modern Roman system. If we needed additional evidence that the Romans originally employed natural number-signs, we might cite the passage in Livy VII, 3, where we are told, that, in accordance with a very ancient law, a nail was annually driven into a certain spot in the sanctuary of

* *System der Arithmetik.* (Potsdam : Aug. Stein. 1885.)

Minerva, the "inventrix" of counting, for the purpose of showing the number of years which had elapsed since the building of the edifice. We learn from the same source that also in the temple at Volsinii nails were shown which the Etruscans had placed there as marks for the number of years.

Also recent researches in the civilisation of ancient Mexico show that natural number-pictures were the first stage of numeral notation. Whosoever has carefully studied in any large ethnographical collection the monuments of ancient Mexico, will surely have remarked that the nations which inhabited Mexico before its conquest by the Spaniards, possessed natural number-signs for all numbers from one to nineteen, which they formed by combinations of circles. If in our studies of the past of modern civilised peoples, we meet with natural number-pictures only among the Greeks or Romans, and some Oriental nations, the reason is that the other nations, as the Germans, before they came into contact with the Romans and adopted the more highly developed notation of the latter, were not yet sufficiently advanced in civilisation to feel any need of expressing numbers symbolically. But since the most perfect of all systems of numeration, the Hindu system of "local value," was introduced and adopted in Europe in the twelfth century, the Roman numeral system gradually disappeared, at least from practical computation, and at present we are only reminded by the Roman characters of inscriptions of the first and primitive stage of all numeral notation. Today we see natural number-pictures, except in the above-mentioned games, only very rarely, as where the tally-men of wharves or warehouses make single strokes with a pencil or a piece of chalk, one for each bale or sack which is counted.

As in writing it is of consequence to associate with each of the things to be counted some simple sign, so in speaking it is of consequence to utter for each single thing counted some short sound. It is quite indifferent here what this sound is called, also, whether the sounds which are associated with the things to be counted are the same in kind or not, and finally, whether they are uttered at equal or unequal intervals of time. Yet it is more convenient and simpler to employ the same sound and to observe equal intervals in

their utterance. We arrive thus at natural number-words. For example, utterances like,

oh, oh-oh, oh-oh-oh, oh-oh-oh-oh, oh-oh-oh-oh-oh,

are natural number-words for the numbers from one to five. Number-words of this description are not now to be found in any known language. And yet we hear such natural number-words constantly, every day and night of our lives; the only difference being that the speakers are not human beings but machines—namely, the striking-apparatus of our clocks.

Word-forms of the kind described are too inconvenient, however, for use in language, not only for the speaker, on account of their ultimate length, but also for the hearer, who must be constantly on the *qui vive* lest he misunderstand a numeral word so formed. It has thus come about that the languages of men from time immemorial have possessed numeral words which exhibit no trace of the original idea of single association. But if we should always select for every new numeral word some new and special verbal root, we should find ourselves in possession of an inordinately large number of roots, and too severely tax our powers of memory. Accordingly, the languages of both civilised and uncivilised peoples always construct their words for larger numbers from words for smaller numbers. What number we shall begin with in the formation of compound numeral words is quite indifferent, so far as the idea of number itself is concerned. Yet we find, nevertheless, in nearly all languages one and the same number taken as the first station in the formation of compound numeral words, and this number is ten. Chinese and Latins, Fins and Malays, that is, peoples who have no linguistic relationship, all exhibit in the formation of numeral words the similarity of beginning with the number ten the formation of compound numerals. No other reason can be found for this striking agreement than the fact that all the forefathers of these nations possessed ten fingers.

Granting it were impossible to prove in any other way that people originally used their fingers in reckoning, the conclusion could be inferred with sufficient certainty solely from this agreement with regard to the first resting-point in the formation of compound

numerals among the most various races. In the Indo-Germanic tongues the numeral words from ten to ninety-nine are formed by composition from smaller numeral words. Two methods remain for continuing the formation of the numerals: either we take a new root as our basis of composition (hundred) or we go on counting from ninety-nine, saying tenty, eleventy, etc. If we were logically to follow out this second method we should get tenty-ty for a thousand, tenty-ty-ty for ten thousand, etc. But in the utterance of such words, the syllable *ty* would be so frequently repeated that the same inconvenience would be produced as above in our individual number-pictures. For this reason the genius which controls the formation of speech took the first course.

But this course is only logically carried out in the old Indian numeral words. In Sanskrit we not only have for ten, hundred, and thousand a new root, but new bases of composition also exist for ten thousand, one hundred thousand, ten millions, etc., which are in no wise related with the words for smaller numbers. Such roots exist among the Hindus for all numerals up to the number expressed by a one and fifty-four appended naughts. In no other language do we find this principle carried so far. In most languages the numeral words for the number consisting of a one with four and five appended naughts are compounded, and in further formations use is made of the words million, billion, trillion, etc., which really exhibit only one root, before which numeral words of the Latin tongue are placed.

Besides numeral word-systems based on the number *ten*, logical systems are only found based on the number five and on the number twenty. Systems of numeral words which have the basis five occur in equatorial Africa. (See the language-tables of Stanley's books on Africa.) The Aztecs and Mayas of ancient Mexico had the base twenty. In Europe it was mainly the Celts who reckoned with twenty as base. The French language still shows some few traces of the Celtic vicenary system, as in its word for eighty, *quatre-vingt*. The choice of five and twenty as bases is explained simply enough by the fact that each hand has five fingers, and that hands and feet together have twenty fingers and toes.

As we see, the languages of humanity now no longer possess natural number-signs and number-words, but employ names and systems of notation adopted subsequently to this first stage. Accordingly, we must add to the definition of counting above given a third factor or element which, though not absolutely necessary, is yet important, namely, that we must be able to express the results of the above-defined associating of certain other things with the things to be counted, by some conventional sign or numeral word.

Having thus established what counting or *numbering* means, we are in a position to define also the notion of *number*, which we do by simply saying that by number we understand *the results* of counting or numeration, which are naturally composed of two elements. First, of the ordinary number-word or number-sign; and secondly, of the word standing for the specific things counted. For example, eight men, seven trees, five cities. When, now, we have counted one group of things, and subsequently also counted another group of things of the same kind, and thereupon we conceive the two groups of things combined into a single group, we can save ourselves the labor of counting the things a third time by blending the number-pictures belonging to the two groups into a single number-picture belonging to the whole. In this way we arrive on the one hand at the idea of addition, and on the other, at the notion of "unnamed" number. Since we have no means of telling from the two original number-pictures and the third one which is produced from these, the kind or character of the things counted, we are ultimately led in our conception of number to abstract wholly from the nature of the things counted, and to form the definition of unnamed number.

We thus see that to ascend from the notion of named number to the notion of unnamed number, the notion of addition is necessary, joined to a high power of abstraction. Here again our theory is best verified by observations of children learning to count and add. A child, in beginning arithmetic, can well understand what five pens or five chairs are, but he cannot be made to understand from this alone what five abstractly is. But if we put beside the first five pens three other pens, or beside the five chairs three other

chairs, we can usually bring the child to see that five things plus three things are always eight things, no matter of what nature the things are, and that accordingly we need not always specify in counting what kind of things we mean. At first we always make the answer to our question of what five plus three is, easy for the child, by relieving him of the process of abstraction, which is necessary to ascend from the named to the unnamed number, an end which we accomplish by not asking first what five plus three is, but by associating with the numbers words designating things within the sphere of the child's experience, for example, by asking how many five pens plus three pens are.

The preceding reflexions have led us to the notion of unnamed or abstract numbers. The arithmetician calls these numbers positive whole numbers, or positive integers, as he knows of other kinds of numbers, for example, negative numbers, irrational numbers, etc. Still, observation of the world of actual facts, as revealed to us by our senses, can naturally lead us only to positive whole numbers, such only, and no others, being results of actual counting. All other kinds of numbers are nothing but artificial inventions of mathematicians created for the purpose of giving to the chief tool of the mathematician, namely, arithmetical notation, a more convenient and more practical form, so that the solution of the problems which arise in mathematics may be simplified. All numbers, excepting the results of counting above defined, are and remain mere symbols, which, although they are of incalculable value in mathematics, and, therefore, can scarcely be dispensed with, yet could, if it were a question of principle, be avoided. Kronecker has shown that any problem in which positive whole numbers are given, and only such are sought, always admits of solution without the help of other kinds of numbers, although the employment of the latter wonderfully simplifies the solution.

How these derived species of numbers, by the logical application of a single principle, naturally flow from the notion of number and of addition above deduced, I shall show in a subsequent article entitled "Monism in Arithmetic."

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